Lesson 22: Linear Inequalities, Absolute Value

Linear Inequalities and Interval Notation

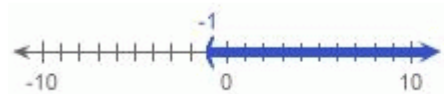
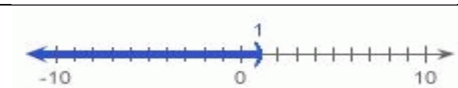
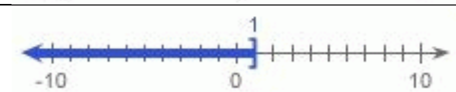
Solving Linear Inequalities in One Variable

When the equals sign in a linear equation is replaced with one of the inequality symbols (<, >, ≤, or ≥) a **linear inequality** is formed.

A major difference between a linear equation and a linear inequality is that the **solution set** of the linear inequality may have an unlimited number of elements.

Graphing and Interval Notation for Strict and Weak Inequalities

There are at least two ways to represent the solution to an inequality: **Graphing** and **interval notation**.

Inequality	Solution Set with Interval Notation	Graph
$x > -1$	$(-1,\ \infty)$	
$x \geq -6$	$[-6,\ \infty)$	
$x < 1$	$(-\infty, 1)$	
$x \leq 1$	$(-\infty, 1]$	
	Hint: In interval notation, the smallest value must come first.	Hint: When the end value IS included, use the] .

A **strict** inequality uses **<** or **>** and the graph's end point is represented by a (or).
A **weak** inequality uses ≤ or ≥ and the graph's end point is represented by [or].

Consider: $x < 3$

Graph it:

Give the Interval notation:

Consider: $x \geq -4$

Graph it:

Give the Interval notation:

Equivalent and Linear Inequalities

Equivalent inequalities: Inequalities with the same solution set are called equivalent inequalities.

$x < 3$ and $x + 2 < 5$ are equivalent inequalities.

Linear Inequality: A linear inequality in one variable x is any inequality of the form $ax + b < 0$, where a and b are real numbers, with $a \neq 0$. [In place of < one may also use >, ≤, or ≥.]

Linear Inequalities: Addition and Multiplication Properties

Addition Property of Inequality: If the same number is added to both sides of an inequality, then the solution set to the inequality is unchanged. For any algebraic expressions A, B, and C, if A < B, then A + C < B + C
[Note: No change in direction of inequality symbol]

$x-5<8$
$x-5+5<8+5$
$x<13$
$\{x \mid x<13\}$ Interval Notation: $(-\infty,13)$
$7-x\le 9$
$7-x+x\le 9+x$
$7\le 9+x$
$7-9\le 9-9+x$
$-2\le x$
$\{x \mid -2\le x\}$ or $\{x \mid x\ge -2\}$ Either way, the x is greater than or equal to -2.
$[-2,\infty)$ The bracket indicates 'or equal to' -2.

TRY: $x+6<15$ $5+x\ge 10$

Multiplication Property of Inequality: If both sides of an inequality are multiplied by the same **positive** number, then the solution set to the inequality is unchanged. For any algebraic expressions A, B, and C, if A < B and C > 0 (positive), then A • C < B • C
[Note: No change in direction of inequality symbol]

Examples:

$\frac{1}{2}x\le 10$
$2\cdot\frac{1}{2}x\le 10\cdot 2$
$x\le 20$
$\{x \mid x\le 20\}$
$(-\infty,20]$

$5x>15$
$\frac{1}{5}\cdot 5x>15\cdot\frac{1}{5}$ also thought of as $\frac{5x}{5}>\frac{15}{5}$
$x>3$
$\{x \mid x>3\}$
$(3,\infty)$

TRY: $\frac{1}{4}x \geq 8$ $4x < 8$

Multiplication Property of Inequality: If both sides of an inequality are multiplied by the same **negative** number and the inequality symbol is reversed, then the solution set to the inequality is unchanged. For any algebraic expressions A, B, and C, if A < B and C < 0 (negative), then A • C > B • C

[Note: Direction of inequality symbol is reversed]

	$-x > 3$	$-5x \geq 35$	$8 > -\frac{x}{4}$
	$-x \cdot (-1) > 3 \cdot (-1)$	$-5x \cdot \left(-\frac{1}{5}\right) \geq 35 \cdot \left(-\frac{1}{5}\right)$	$8 \cdot (-4) > -\frac{x}{4} \cdot (-4)$
SIGN REVERSED	$x < -3$	$x \leq -7$	$-32 < x$
	$\{x \mid x < -3\}$ $(-\infty, -3)$	$\{x \mid x \leq -7\}$ $(-\infty, -7]$	$\{x \mid x > -32\}$ putting x first $(-32, \infty)$

TRY: $-\frac{1}{3}x < 9$ $-8x \leq 16$

One can multiply and divide both sides of an inequality by the same positive expression and maintain the direction of the inequality symbol. But, <u>if one multiplies or divides both sides of an inequality by the same negative expression, the direction of the inequality symbol must be reversed</u>.

Solve each of the following inequalities. Express the solution set in interval notation and graph it.

$$3x-2<6$$
$$3x-2+2<6+2$$
$$3x<8$$
$$\frac{3x}{3}<\frac{8}{3}$$
$$x<\frac{8}{3}$$
$$\{x \mid x<\frac{8}{3}\}$$
$$(-\infty,\frac{8}{3})$$

$$19\le 5-4x$$
$$19-5\le 5-5-4x$$
$$14\le -4x$$
$$\frac{14}{-4}\le\frac{-4x}{-4}$$

Be sure to FLIP the sign!

$$-\frac{7}{2}\ge x$$
$$x\le -\frac{7}{2}$$
$$\{x \mid x\le -\frac{7}{2}\}$$
$$(-\infty,-\frac{7}{2}]$$

$$2x+3>2(x-4)$$
$$2x+3>2x-8$$
$$2x+3-3>2x-8-3$$
$$2x>2x-11$$
$$2x-2x>2x-2x-11$$
$$0>-11$$

This is a TRUE statement. Therefore, the entire number line is the solution.

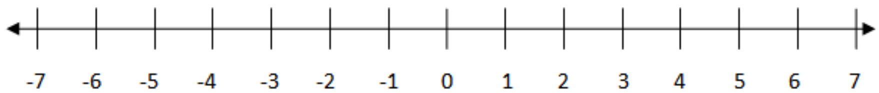
$$\{x \mid x\in \mathrm{Re}\,als\}$$
$$(-\infty,\infty)$$

TRY: 3x – 6 <15   7 ≤ 5 – 3x   3x + 2 > -5+ 3x

Solving Absolute Value Equations

Absolute Value Review

The absolute value of a number represents the **undirected distance** from that number to the origin on the number line, that is, the distance from x to 0.



The absolute value does not depend on whether the number is to the left or right of 0. It only depends on its distance from 0. The absolute value of a number a is written $|a|$.

(EXTRA) Examples:

$|5| = 5$ $\qquad$ $|-3| = 3$ $\qquad$ $|0| = 0$ $\qquad$ $-|-8| = -8$

$|-6| + |4| =$
$6 + 4 = 10$

$|-14| - |-11| =$
$14 - 11 = 3$

Another way of thinking: Visualize yourself standing on the number line. If you were standing at 5, it would take you 5 steps to reach 0. The absolute value of 5, or |5|, is 5. If you were at -3, it would take you 3 steps to reach 0. The absolute value of -3, or |-3|, is 3. If you were at 0, it would take 0 steps to reach 0. The absolute value of 0, or |0|, is 0.

Basic Absolute Value Equations

Absolute Value Equation	Equivalent Equation	Solution Set
$\lvert x \rvert = k$ where $k > 0$	$x = k$ or $x = -k$	$\{k, -k\}$
$\lvert x \rvert = 0$	$x = 0$	$\{0\}$
$\lvert x \rvert = k$ where $k < 0$		$\varnothing$

Steps for Solving Absolute Value Equations:

1. Isolate the absolute value on one side of the equation.
2. Write the **two** equations that are equivalent to the absolute value equation.
3. Solve each equation.
4. Check each answer in the original absolute value equation.

$\lvert x \rvert = 5$ $x = -5$ OR $x = 5$ $\{-5, 5\}$ List solutions in numerical order.	$\lvert x \rvert = -6$ No solution or $\varnothing$ Absolute value cannot be negative.	$\lvert x - 7 \rvert = 3$ $x - 7 = -3$ OR $x - 7 = 3$ $x - 7 + 7 = -3 + 7$ $x - 7 + 7 = 3 + 7$ $x = 4$ $x = 10$ $\{4, 10\}$

TRY: $\lvert x \rvert = 2$ $\lvert 7 - x \rvert = 15$

$\lvert 5x + 2 \rvert = -3$ $\lvert 3 - 3x \rvert = 4$

In instances where the absolute value is NOT isolated, isolate it first on one side, then solve.

DO NOT determine if the equation can be solved or not until the absolute value is by itself.

$4-3\|x-2\|=-8$ $4-4-3\|x-2\|=-8-4$ $\frac{-3\|x-2\|}{-3}=\frac{-12}{-3}$ $\|x-2\|=4$ $x-2=-4$ or $x-2=4$ $x-2+2=-4+2$ or $x-2+2=4+2$ $x=-2$ or $x=6$ $\{-2,6\}$	Subtract 4 from each side to isolate the term with the absolute value. Divide both sides by -3 to isolate the absolute value. Solve as before.

TRY: 5 - 4| x+3 | = -43 (EXTRA) | x+3 |- 5 = 7 Did you get {-15,9}?

Double Absolute Value

Treat this one as if the right side was not an absolute value. | w – 6 | = | 3 – 2w |

Try:

Think of it as | w – 6 | = 3 – 2w

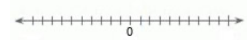
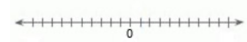
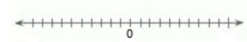
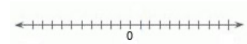
Then set it up as: w – 6 = + (3 – 2w) or w – 6 = – (3 – 2w) and solve each part

Solving Absolute Value Inequalities

Absolute Value Inequalities

$|x| \le 2$ is an example of an **absolute value inequality**.

Basic Absolute Value **Inequality** $(k > 0)$

Absolute Value Inequality	Equivalent Inequality	Solution Set	Graph (for you to do)
$\lvert x \rvert > k$	$x > k$ or $x < -k$	$(-\infty, -k) \cup (k, \infty)$	
$\lvert x \rvert \ge k$	$x \ge k$ or $x \le -k$	$(-\infty, -k] \cup [k, \infty)$	
$\lvert x \rvert < k$	$-k < x < k$	$(-k, k)$	
$\lvert x \rvert \le k$	$-k \le x \le k$	$[-k, k]$	

$|x| \le 2$ is equivalent to $-2 \le x \le 2$ The solution set is $[-2, 2]$

Compound Inequalities

The Equivalent Inequalities shown in the chart above are examples of **compound inequalities**.

If one joins two simple inequalities with the connective "or" or the connective "and", one gets a **compound inequality**.

$x < -8$ or $x > 8$ — A compound inequality using the connective "OR" is true if one **or** the other **or** both of the simple inequalities are true. It is false only if both simple inequalities are false. The solution set to an “OR” compound inequality consists of all numbers that satisfy at least one of the simple inequalities (the union of the numbers).
$(-\infty, -8) \cup (8, \infty)$

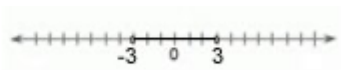
$x > -8$ and $x < 8$ — (often written in the form $-8 < x < 8$)
A compound inequality using the connective "AND" is true if and only if **both** simple inequalities are true. The solution set to an “AND” compound inequality consists of all numbers that satisfy both simple inequalities (the intersection of the numbers).
$(-8, 8)$

When working in word problems, the word **between** translates into a compound inequality using strict inequality symbols < or >.

Words **from** and **to** translates into weak inequality symbols ≤ or ≥.

Use compound inequalities to solve these Basic Absolute Value Inequalities.

$| w | < 3$



$-3 < w$ and $w < 3$

$-3 < w < 3$

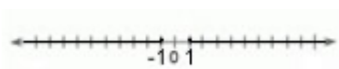
$(-3, 3)$

First, isolate the absolute value. (It is.)

Next, draw a number line marking 0, the value and its opposite. That is, mark the 3 and -3.
Since this is a less than, indicate the area of the line between 0 and 3 that is represented by the statement: $w < 3$.
Now, indicate the opposite area … the area between 0 and -3.
This represents an AND.

Rewrite the problem into two parts and solve.

$| w - 5 | \geq 1$



$w - 5 \leq -1$ or $w - 5 \geq 1$

$w - 5 + 5 \leq -1 + 5$ or $w - 5 + 5 \geq 1 + 5$

$w \leq 4$ or $w \geq 6$

$(-\infty, 4] \cup [6, \infty)$

First, isolate the absolute value.

Next, draw a number line marking 0, the value and its opposite. That is, mark the 1 and -1.
Since this is a greater than, indicate the area greater than 1 that is represented by w-5.
Now, indicate the opposite area … the area less than -1.

This represents an OR. Rewrite the problem into two parts and solve.

The final answer cannot be combined. It is written as the union of two intervals.

TRY: $| 3x | < 15$

$4 \leq | x | - 6$

$- 3 \mid 6 - x \mid \geq - 3$

$| 6 - x | \geq 0$