Lesson 20: Rationalizing the Denominator & Complex Numbers

Rationalizing the Denominator: One Square Root

Standard Form for a Radical Expression is when…

1. The radicand contains no factors that can be written to an exponent greater than or equal to the index. $\sqrt[3]{b^4}$ *is not in standard form*.
2. The exponent of each factor of the radicand and the index of the radical have no common factor other than the number 1. $\sqrt[6]{b^3}$ *is not in standard form*.
3. The radicand contains no fractions. $\sqrt[3]{\frac{a}{b}}$ *is not in standard form*.
4. No radicals appear in the denominator. $\frac{1}{\sqrt{b}}$ *is not in standard form*.

Rationalizing the Denominator:

the process of eliminating radicals from the denominator of an expression

Procedure for Rationalizing a Denominator of One Term

1. Multiply the numerator and the denominator by a radical with the same index as the radical that one wants to eliminate from the denominator.

2. The exponent of each factor of the radicand must be such that the product of the radicands in the denominator results in a radical that is a perfect *n*th root.

3. Carry out the multiplication and reduce the fraction if possible.

   Don't forget to simplify the inside of the radical first if possible.

With a square root, $\frac{5}{2x\sqrt{3}}$, one needs a total of 2 factors of 3 under the radical in the denominator, so multiply the numerator and denominator by another $\sqrt{3}$.

$$\frac{5}{2x\sqrt{3}} = \frac{5}{2x\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{2x\sqrt{9}} = \frac{5\sqrt{3}}{2x \cdot 3} = \frac{5\sqrt{3}}{6x}$$

$$\frac{\sqrt{6}}{\sqrt{5}} = \frac{\sqrt{6}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{30}}{\sqrt{25}} = \frac{\sqrt{30}}{5}$$

TRY:

$$\frac{2}{3\sqrt{a}} \qquad\qquad \frac{\sqrt{3}}{\sqrt{7}}$$

$\frac{\sqrt{3}}{\sqrt{18}}$ Since 3 and 18 have common factors, rewrite the radicand as one fraction and simplify first.

$$\frac{\sqrt{3}}{\sqrt{18}} = \sqrt{\frac{3}{18}} = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

If the radicand in the denominator can be simplified first, do so if desired.

$$\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{\sqrt{4 \cdot 2}} = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2\sqrt{4}} = \frac{\sqrt{6}}{2 \cdot 2} = \frac{\sqrt{6}}{4}$$

One arrives at the same result by not reducing first.

$$\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{24}}{\sqrt{64}} = \frac{\sqrt{4 \cdot 6}}{8} = \frac{\sqrt{4}\sqrt{6}}{8} = \frac{2\sqrt{6}}{8} = \frac{\sqrt{6}}{4}$$

Either way, one reduces at some point in the process.

TRY:

$$\frac{5\sqrt{5}}{4\sqrt{12}} \qquad\qquad \sqrt{\frac{24m^5}{6m^3}}$$ (simplify inside first)

Rationalizing the Denominator: Higher Root

When rationalizing a denominator containing a term that has a root higher than 2, multiply the numerator and the denominator by the value needed to form a perfect cube, 4th, 5th, etc. of the denominator.

$\frac{7}{\sqrt[3]{x}}$ has a cube root. Therefore, 3 factors of x are needed in the radicand. $\frac{7}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{7\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{7\sqrt[3]{x^2}}{x}$

$\sqrt[3]{\frac{3}{4a^2}}$ Cannot be simplified, so split it. $\frac{\sqrt[3]{3}}{\sqrt[3]{4a^2}}$ Need 3 factors of 2 and 1 factor of a in the radicand.

$$\frac{\sqrt[3]{3}}{\sqrt[3]{4a^2}} = \frac{\sqrt[3]{3}}{\sqrt[3]{4a^2}} \cdot \frac{\sqrt[3]{2a}}{\sqrt[3]{2a}} = \frac{\sqrt[3]{6a}}{\sqrt[3]{8a^3}} = \frac{\sqrt[3]{6a}}{2a}$$

$$\sqrt[5]{\frac{3}{x^2 y}} = \frac{\sqrt[5]{3}}{\sqrt[5]{x^2 y}} \cdot \frac{\sqrt[5]{x^3 y^4}}{\sqrt[5]{x^3 y^4}} = \frac{\sqrt[5]{3x^3 y^4}}{\sqrt[5]{x^5 y^5}} = \frac{\sqrt[5]{3x^3 y^4}}{xy}$$

TRY:

$\sqrt[3]{\frac{3}{5}}$

$\sqrt[4]{\frac{2}{27x^2}}$

$\frac{9}{\sqrt[5]{a^3 b^4}}$

$\sqrt[3]{\frac{4a}{b}}$

Rationalizing the Denominator: Two Terms

Multiply the numerator and the denominator by the conjugate of the denominator.

$$\frac{6}{3-\sqrt{3}} = \frac{6}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}} = \qquad \frac{18+6\sqrt{3}}{9-\sqrt{9}} = \frac{18+6\sqrt{3}}{9-3} = \frac{6\left(3+\sqrt{3}\right)}{6} = 3+\sqrt{3}$$

$$\frac{2-\sqrt{3}}{\sqrt{2}+\sqrt{6}} = \frac{2-\sqrt{3}}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}} = \frac{2\sqrt{2}-2\sqrt{6}-\sqrt{6}+\sqrt{18}}{\sqrt{4}-\sqrt{36}} = \frac{2\sqrt{2}-3\sqrt{6}+\sqrt{9\cdot 2}}{2-6} =$$

$$\frac{2\sqrt{2}-3\sqrt{6}+3\sqrt{2}}{-4} = -\frac{5\sqrt{2}-3\sqrt{6}}{4}$$

$$\frac{5}{x+\sqrt{7}} = \frac{5}{x+\sqrt{7}}\cdot\frac{x-\sqrt{7}}{x-\sqrt{7}} = \frac{5(x-\sqrt{7})}{x^2-7} = \frac{5x-5\sqrt{7})}{x^2-7}$$

TRY:

$$\frac{5}{\sqrt{7}-\sqrt{5}} \qquad\qquad \frac{8}{x+\sqrt{6}}$$

Some expression can be simplified first by dividing out common factors from the numerator and the denominator.

$$\frac{21+14\sqrt{2}}{7} = \frac{7\left(3+2\sqrt{2}\right)}{7} = 3+2\sqrt{2}$$

Some expressions can be simplified first before rationalizing.

$\frac{5}{10+\sqrt{50}} = \frac{5}{10+\sqrt{25\cdot 2}} = \frac{5}{10+5\sqrt{2}} = \frac{5}{5(2+\sqrt{2})} = \frac{1}{2+\sqrt{2}}$ Now, rationalize the denominator.

$$\frac{1}{2+\sqrt{2}} = \frac{1}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{2-\sqrt{2}}{4-\sqrt{4}} = \frac{2-\sqrt{2}}{4-2} = \frac{2-\sqrt{2}}{2}$$

$\frac{6}{3-3\sqrt{2}} = \frac{6}{3(1-\sqrt{2})} = \frac{2}{1-\sqrt{2}}$ Now, rationalize the denominator.

$$\frac{2}{1-\sqrt{2}} = \frac{2}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{4}} = \frac{2+2\sqrt{2}}{1-2} = \frac{2+2\sqrt{2}}{-1} = -(2+2\sqrt{2}) = -2-2\sqrt{2}$$

TRY:

$$\frac{6}{4-4\sqrt{2}}$$

$$\frac{x-16}{\sqrt{x}-4}$$

Complex Numbers

Solve $x^2+9=0$ using the square root property.

$x^2+9=0 \qquad x^2=-9 \qquad \sqrt{x^2}=\sqrt{-9} \qquad x=\pm\sqrt{-9}$

But the square root of a negative number is not real.
Welcome to the world of imaginary numbers!

Definition of *i*

The number *i* is a number such that $i=\sqrt{-1}$ and $i^2=-1$

These are important to remember!

A **complex number** is a number of standard form $a+bi$,
where a and b are real numbers and $i=\sqrt{-1}$.
a Is called the real part and b is called the imaginary part.

All Reals are a subset of the complex numbers as 7 could be written: $7+0i$

$3i$ could be written $0+3i$. Written as $3i$, it is considered a pure imaginary number.

Finding the square root of a Negative Number

$$\sqrt{-25}=\sqrt{-1\cdot 25}=i\sqrt{25}=5i \qquad \sqrt{-75}=\sqrt{-1\cdot 25\cdot 3}=i\sqrt{25\cdot 3}=5i\sqrt{3}$$

$$\sqrt{-20}=\sqrt{-1\cdot 4\cdot 5}=i\sqrt{4\cdot 5}=2i\sqrt{5}$$

Notice the placement of the values in the answer. To maintain the proper form for the complex number, the number of the imaginary part (in this case the 5) , is placed in front of i. To avoid confusion of thinking the $5i$ is under the radical, the imaginary part is always placed in front of any radical sign.

TRY:

$\sqrt{-36} \qquad \sqrt{-49} \qquad \sqrt{-32} \qquad \sqrt{-300}$

Multiplying Square Roots of Negative Numbers

Multiply and simplify: $\sqrt{18}\cdot\sqrt{2}$ $\sqrt{18}\cdot\sqrt{2}=\sqrt{36}=6$

Now multiply and simplify: $\sqrt{-18}\cdot\sqrt{-2}$ does it also equal $\sqrt{36}$? **NO!**

In the imaginary world, multiplication of negative values works differently!!!
So what does it equal?

To multiply square roots of negative numbers, express the radical in terms of $\sqrt{-18}\cdot\sqrt{-2}$,

first rewrite the values in terms of i before proceeding. $\sqrt{-1\cdot 18}\cdot\sqrt{-1\cdot 2}=i\sqrt{18}\cdot i\sqrt{2}$

Next, multiply the i's , then multiply the radicals. Remember, $i^2=-1$.

$$\sqrt{-1\cdot 18}\cdot\sqrt{-1\cdot 2}=i\sqrt{18}\cdot i\sqrt{2}=i^2\sqrt{36}=-1\cdot 6=-6$$

It is a very common error to forget the negative!

$$\sqrt{-7}\cdot\sqrt{-11}=i\sqrt{7}\cdot i\sqrt{11}=i^2\sqrt{77}=-\sqrt{77}$$

TRY:

$\sqrt{-3}\cdot\sqrt{-6}$ $\sqrt{-5}\cdot\sqrt{-20}$

Adding and Subtracting Complex Numbers

To add or subtract complex numbers, combine the real parts together and combine the imaginary parts together.

$(5+7i)+(6-2i)=11+5i$ $(5+7i)-(6-2i)=5+7i-6+2i=-1+9i$

TRY:

$(3+4i)+(5-6i)$ $(7-6i)-(4-2i)$

Multiplying Complex Numbers

To multiply complex numbers, multiply in the same way one multiplies polynomials. However, be sure to remember to replace i^2 with -1.

Monomial times a binomial: $3i(4i+5)=12i^2+15i=-12+15i$

TRY: $-5i(3i-4)$

Two binomials: $(2i+4)(3i-5)=6i^2-10i+12i-20=-6+2i-20=-26+2i$

Use FOIL to multiply two complex numbers together. Be sure to express the answer in proper form.

TRY: $(3i+4)(2i-6)$

The conjugate of $(a+bi)$ is $(a-bi)$. $(6+2i)$ and $(6-2i)$ are examples of **complex conjugates**.

Just like multiplying together two conjugates with radicals "rationalizes" the result and removes the radical, multiplying two complex conjugates together removes the imaginary part and produces a "real" result.

$$(6+2i)(6-2i)=36-12i+12i-4i^2=36-4i^2=36-(-4)=36+4=40$$

TRY: Multiply this complex number and its conjugate together. $(3-5i)$

Dividing Complex Numbers

To divide by a complex number, write the expression as a fraction and multiply both the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{5}{2-3i}=\frac{5}{2-3i}\cdot\frac{2+3i}{2+3i}=\frac{10+15i}{4-9i^2}=\frac{10+15i}{13}$$

TRY: $\dfrac{-4}{3-2i}$

