Lesson 19: Working with Roots and Radicals

Multiplying Roots

Product Rule for Square Roots

If a and b are nonnegative real numbers, then $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$

$$\sqrt{7} \cdot \sqrt{2y} = \sqrt{14y} \qquad \sqrt{5} \cdot \sqrt{7} = \sqrt{35}$$

This type of multiplication of radicals is only possible if the indices (roots) are the same.

TRY: $\sqrt{5} \cdot \sqrt{11}$ $\qquad$ $\sqrt{7} \cdot \sqrt{3x}$

Root Chart $\sqrt{a} = b$

b ➔	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sqrt{a}$	4	9	16	25	36	49	64	81	100	121	144	169	196	225

Square Roots Simplified

An expression containing a square root is simplified if:

- The radicand does not contain any factors that are perfect squares other than 1.
- The radicand does not contain any variables with exponents greater than 1.
- No radicals remain in the denominator of a fraction.

Square Root of a Whole Number

To simplify a radical containing a radicand that is not a perfect square, reverse the Product Rule for Square Roots. Rewrite the radicand as the product of factors where as many factors as possible are perfect squares. Write each factor as a separate radical and evaluate each. Multiply the results.

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b} \qquad \sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$$

$$\sqrt{500} = \sqrt{100 \cdot 5} = \sqrt{100} \cdot \sqrt{5} = 10\sqrt{5}$$

This process helps when one doesn't recognize larger perfect squares. Factor the radicand completely. (Use a factor tree if necessary.) Group two similar factors together to form perfect squares.

Example: (The thinking process for factoring 1764.)

$\sqrt{1764}$

1764 = 2 · 882	(1764 - even – must be divisible by 2.)
1764 = 2 · 2 · 441	(882 – even – must be divisible by 2.)
1764 = 2 · 2 · 3 · 147	(441 – digits add up to a multiple of 3, so divisible by 3.)
1764 = 2 · 2 · 3 · 3 · 49	(147 – digits add up to a multiple of 3, divisible by 3.)
1764 = 2 · 2 · 3 · 3 · 49	(recognize 49 as a perfect square, combine like factors)
1764 = 4 · 9 · 49	

$$\sqrt{1764} = \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{49} = 2 \cdot 3 \cdot 7 = 42$$

If the original had been $\sqrt{3528}$, the number 3528 would have factored into 2 · 2 · 2 · 3 · 3 · 49.
The result would have been: $\sqrt{3528} = \sqrt{2} \cdot \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{49} = \sqrt{2} \cdot 2 \cdot 3 \cdot 7 = 42\sqrt{2}$

TRY:

$\sqrt{45}$ $\qquad$ $\sqrt{72}$

$\sqrt{98}$ $\qquad$ $\sqrt{125}$

Square Root of a Variable

If a is a positive real number and *m* is an even integer, then $\sqrt{a^m} = a^{\frac{m}{2}}$ Remember: $a^{\frac{power}{root}}$

$\sqrt{a^6} = a^{\frac{6}{2}} = a^3$

$\sqrt{z^{36}} = z^{\frac{36}{2}} = z^{18}$ CAUTION: It is very tempting to take the square root of 36and answer incorrectly z^6

If *m* is an odd integer, rewrite *m* as the sum of a multiple of 2 and 1.

$\sqrt{a^7} = \sqrt{a^{6+1}} = \sqrt{a^6}\sqrt{a} = a^3\sqrt{a}$

$\sqrt{a^{15}} = \sqrt{a^{14+1}} = \sqrt{a^{14}}\sqrt{a} = a^7\sqrt{a}$

TRY:

$\sqrt{n^5}$

$\sqrt{z^9}$

Combinations: $\sqrt{12x^{11}} = \sqrt{3 \cdot 4 \cdot x^{10} \cdot x} = 2x^5\sqrt{3x}$

TRY:

$\sqrt{12x^8}$

$\sqrt{36n^2}$

$\sqrt{3n^3}$

$\sqrt{8z^{16}}$

Quotient Rule for Roots

Quotient Property for Square Roots

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, where b $\neq$ 0, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Concept: The root of a quotient can be written as the root of the numerator divided by the root of the denominator.

$$\sqrt{\frac{144x^2}{36y^2}} = \frac{\sqrt{144x^2}}{\sqrt{36y^2}} = \frac{12x}{6y} = \frac{2x}{y}$$

Sometimes, it helps to combine the radicals as one: $\frac{\sqrt{72}}{\sqrt{2}} = \sqrt{\frac{72}{2}} = \sqrt{36} = 6$

TRY:

$$\sqrt{\frac{9}{144}} \qquad \frac{\sqrt{50}}{\sqrt{2}}$$

$$\sqrt{\frac{8}{81}} \qquad \sqrt{\frac{9a^2}{49b^4}}$$

Multiplying Higher Roots

Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer greater than 1, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

Concept: The nth root of a product is equal to the product of the nth roots of the factors.

CAUTION: This rule only applies to radicals with the **same** index. $\sqrt[4]{7} \cdot \sqrt[3]{10} \neq \sqrt[?]{70}$

$\sqrt[4]{7} \cdot \sqrt[4]{10} = \sqrt[4]{70}$ $\sqrt[4]{7} \cdot \sqrt[4]{5x^2} = \sqrt[4]{35x^2}$

TRY:

$\sqrt[4]{5} \cdot \sqrt[4]{6}$ $\sqrt[4]{8} \cdot \sqrt[4]{5x^3}$

Root Chart $\sqrt[n]{a} = b$

b ➔	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sqrt{a}$	4	9	16	25	36	49	64	81	100	121	144	169	196	225
$\sqrt[3]{a}$	8	27	64	125	216				1000					
$\sqrt[4]{a}$	16	81	256	625					10000					
$\sqrt[5]{a}$	32	243	1024											

Radicals Simplified

An expression $\sqrt[n]{p}$ is simplified if:

- The radicand does not contain any factors, other than 1, that are perfect *n*th powers.
- The radicand does not contain any variables with exponents greater than *n*.
- No radicals remain in the denominator of a fraction.

Simplifying Higher Roots

Simplifying Radicals of Higher Roots $\sqrt[n]{p}$

To simplify a radical containing a radicand that is not a perfect *n*th power, reverse the Product Rule for Radicals. Rewrite the radicand as the product of factors where as many factors as possible are perfect *n*th powers. Write each factor as a separate radical and evaluate each. Multiply the results.

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

$$\sqrt[3]{27000} = \sqrt[3]{27}\cdot\sqrt[3]{1000} = 3\cdot 10 = 30$$

$$\sqrt[3]{250} = \sqrt[3]{125}\cdot\sqrt[3]{2} = 5\sqrt[3]{2}$$

TRY:

$$\sqrt[3]{8000}$$

$$\sqrt[3]{270}$$

Using the factor-tree approach works for higher roots as well. The difference is that one wants to form groups of 'n' like factors. That is, if the index is 3, look for 3 of the same factor; if 4, look for groups of 4 like factors, etc.

$$\sqrt[3]{8640} \qquad 8640 = \boxed{2\cdot 2\cdot 2}\cdot\boxed{2\cdot 2\cdot 2}\cdot\boxed{3\cdot 3\cdot 3}\cdot 5$$

$$\sqrt[3]{8640} = \sqrt[3]{2^3\cdot 2^3\cdot 3^3\cdot 5} = 2\cdot 2\cdot 3\cdot\sqrt[3]{5} = 12\sqrt[3]{5}$$

Simplifying Higher Roots with Variables

$\sqrt[n]{a^m} = a^{m/n} = a^{\frac{power}{root}}$

$\sqrt[4]{m^8} = m^{8/4} = m^2$

$\sqrt[4]{a^{20}} = a^{20/4} = a^5$

TRY:

$\sqrt{y^{36}}$ $\sqrt{m^8}$

$\sqrt[4]{x^4}$ $\sqrt[5]{x^{30}}$

What if the root is not a factor of the power as in $\sqrt[4]{a^{11}}$?

Rewrite the radical using the Product Rules: $\sqrt[n]{ab}$ = $\sqrt[n]{a} \cdot \sqrt[n]{b}$ and $a^m a^n = a^{m+n}$

$$\sqrt[4]{a^{11}} = \sqrt[4]{a^{8+3}} = \sqrt[4]{a^8} \cdot \sqrt[4]{a^3} = a^{8/4} \cdot \sqrt[4]{a^3} = a^2 \sqrt[4]{a^3}$$

Another way of thinking.

To simplify a radical completely, think about how many times the index goes into the exponent of the radicand with how many left over. For example, to simplify $\sqrt[4]{a^{11}}$, *ask how many times 4 goes into 11?*

Answer: 2 times with 3 left over.

$$\sqrt[4]{a^{11}} = \sqrt[4]{a^{8+3}} = \sqrt[4]{a^8} \cdot \sqrt[4]{a^3} = a^2 \sqrt[4]{a^3}$$

The '2' exponent comes out and the '3' leftover stays in.

Or one can think of it as: $\sqrt[4]{a^{11}} = a^{\frac{11}{4}} = a^{2\frac{3}{4}} = a^2 \sqrt[4]{a^3}$

TRY:

$\sqrt[3]{x^8}$ $\sqrt[4]{x^{19}}$ $\sqrt[5]{x^{18}}$

The process works with radicands including integers and variables:

$\sqrt[4]{64x^9}$ = $\sqrt[4]{16}\cdot\sqrt[4]{4}\cdot\sqrt[4]{x^8}\cdot\sqrt[4]{x} = 2x^2\sqrt[4]{4x}$

TRY:

$\sqrt[3]{27z^2}$ $\sqrt[3]{5b^9}$

$\sqrt[3]{270x^5}$ $\sqrt[4]{20000x^7}$

$\sqrt[4]{162b^4}$ $\sqrt[5]{96a^8}$ $\sqrt[3]{48x^3y^8z^7}$

Quotient Property for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, where b ≠ 0, and n is a positive integer greater than 1,

then $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Concept: The *n*th root of a quotient can be written as the *n*th root of the numerator divided by the *n*th root of the denominator.

$$\sqrt{\frac{144x^2}{36y^2}} = \frac{\sqrt{144x^2}}{\sqrt{36y^2}} = \frac{12x}{6y} = \frac{2x}{y}$$

$$\frac{\sqrt{72}}{\sqrt{2}} = \sqrt{\frac{72}{2}} = \sqrt{36} = 6$$

TRY:

$$\sqrt[3]{\frac{a^3b^4}{125}}$$

$$\sqrt[4]{\frac{x^5y^4}{z^{12}}}$$

$$\sqrt[4]{\frac{a^7b}{81c^{16}}}$$

$$\sqrt[3]{\frac{-27y^{36}}{1000}}$$

Different Indices

To multiply radicals with different indices, convert the radicals into exponential notation then multiply (work with the exponents). Convert the answer back into a radical if desired.

$$\sqrt{7}\cdot\sqrt[4]{7} = 7^{\frac{1}{2}}7^{\frac{1}{4}} = 7^{\frac{2}{4}}7^{\frac{1}{4}} = 7^{\frac{2}{4}+\frac{1}{4}} = 7^{\frac{3}{4}} = \sqrt[4]{7^3} = \sqrt[4]{343}$$

$$\sqrt[4]{x}\cdot\sqrt[5]{x^2} = x^{1/4}x^{2/5} = x^{1/4+2/5} = x^{5/20+8/20} = x^{13/20} = \sqrt[20]{x^{13}}$$

TRY:

$\sqrt{3}\cdot\sqrt[4]{3}$

$\sqrt[3]{2}\cdot\sqrt[5]{2}$

$\sqrt[3]{3}\cdot\sqrt[4]{2}$

$\sqrt[3]{y^2}\cdot\sqrt[5]{y}$

Radicals: Adding and Subtracting

WARNING ➔ One can only add or subtract **LIKE** radicals.

To Have Like Radicals, the Following Must be True:

1. The radicals must have the same index.
2. The radicals must have the same radicand.

$\sqrt{7}+4\sqrt{7}=5\sqrt{7}$ $\quad\quad$ $6\sqrt[3]{x^2}-2\sqrt[3]{x^2}=4\sqrt[3]{x^2}$

One cannot add radicals with different indices. $\sqrt[3]{7}+4\sqrt{7}$

One cannot add radicals with different radicands. $\sqrt{11}+4\sqrt{7}$

TRY:

$\sqrt{5}-3\sqrt{5}$

$3\sqrt{6a}+7\sqrt{6a}$

$\sqrt[3]{5y}-4\sqrt[3]{5y}+\sqrt[3]{x}+\sqrt[3]{x}$

Sometimes, one needs to simplify the radicals before they can be combined.

$$4\sqrt{75}-2\sqrt{48}$$
$$4\sqrt{25\cdot 3}-2\sqrt{16\cdot 3}$$
$$4\cdot 5\sqrt{3}-2\cdot 4\sqrt{3}$$
$$20\sqrt{3}-8\sqrt{3}$$
$$=12\sqrt{3}$$

$$5x\sqrt[3]{x^5}-2x^2\sqrt[3]{x^2}$$
$$5x\sqrt[3]{x^3\cdot x^2}-2x^2\sqrt[3]{x^2}$$
$$5x^2\sqrt[3]{x^2}-2x^2\sqrt[3]{x^2}$$
$$=3x^2\sqrt[3]{x^2}$$

TRY: $\sqrt{12}+\sqrt{27}$

$$3\sqrt{50}-2\sqrt{32}$$

$$\sqrt[3]{16w^2z^5}-6\sqrt[3]{2w^2z^5}$$

Radicals: Multiplying

To multiply radicals, one often uses the distributive property: $a(b+c)=ab+ac$

The product rule $\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}$ allows multiplication of radicals with the **SAME** index.

$$(\sqrt{5})(-3\sqrt{6})=-3\sqrt{30} \qquad (2\sqrt{7x})(5\sqrt{7})=10\sqrt{49x}=10\cdot 7\sqrt{x}=70\sqrt{x}$$

TRY:

$$3\sqrt{2}\cdot(-4\sqrt{10}) \qquad 2\sqrt{5c}\cdot 5\sqrt{5}$$

To multiply a Binomial containing a radical expression by a Monomial

Use the distributive property.

$\sqrt{3}(5-\sqrt{2})=5\sqrt{3}-\sqrt{6}$ Be sure to simplify the answer if possible.

TRY:

$$7(2-3\sqrt{6}) \qquad -2\sqrt{5}(\sqrt{3}+3\sqrt{5}) \qquad \sqrt{3ab}(\sqrt{3a}+\sqrt{3})$$

To multiply a Binomial containing a radical expression by a Binomial

Use FOIL.

$$(\sqrt{3}+2\sqrt{5})(4-\sqrt{2})=4\sqrt{3}-\sqrt{6}+8\sqrt{5}-2\sqrt{10}$$

TRY:

$$(2\sqrt{6}-3)(2\sqrt{6}+4) \qquad (3\sqrt{2}-\sqrt{3})(2\sqrt{2}+3\sqrt{3}) \qquad (5a+\sqrt{ab})^2$$

Radicals: Conjugates

Conjugates

Consider: $a^2 - b^2 = (a+b)(a-b)$

$(a+b)$ and $(a-b)$ are called **conjugates** of one another.

For example, the conjugate of $5-2\sqrt{3}$ is $5+2\sqrt{3}$

The conjugate of $-3a+\sqrt{7b}$ is $-3a-\sqrt{7b}$

When two conjugates containing radicals are multiplied, the product contains no radicals.

$$(3\sqrt{2x}-4)(3\sqrt{2x}+4) = 9\sqrt{4x^2} + 12\sqrt{2x} - 12\sqrt{2x} - 16 = 9\cdot 2 - 16 = 18 - 16 = 2$$

$$a^2 - b^2 = (3\sqrt{2x})^2 - 4^2 = 9\cdot 2x - 16 = 18x - 16$$

TRY:

$(7-\sqrt{3})(7+\sqrt{3})$ $\qquad$ $(\sqrt{6g}+\sqrt{5})(\sqrt{6g}-\sqrt{5})$ $\qquad$ $(3-2\sqrt{7})(3+2\sqrt{7})$