Lesson 18: Roots, Radicals, Rational Exponents

Definitions: Roots, Radicals

Definition of *n*th Root

If $a = b^n$ for a positive integer *n*, then *b* is the *n*th root of *a*. $\sqrt[n]{a} = b$
If $a = b^2$, then *b* is a square root of *a*. $\sqrt{a} = b$
If $a = b^3$, then *b* is the cube root of *a*. $\sqrt[3]{a} = b$
3 is a second (square) root of 9 since $3^2 = 9$ and $\sqrt{9} = 3$
-2 is the third (cube) root of –8 since $(-2)^3 = -8$ and $\sqrt[3]{-8} = -2$

Radicals

radical symbol $\sqrt[n]{a}$ mathematical sign used to signify roots that is: $\sqrt[index]{radicand}$

a is called the **radicand**, *n* is the **index** (or root) of the radical.
The entire expression $\sqrt[n]{a}$ is called a **radical**.
If there is no index associated with the radical symbol, it is understood to be **2**. Ex: $\sqrt{25}$

Note: If the radicand (the number under the radical sign) is *negative* and the index is *even*, the radical does not represent a real number. Ex: $\sqrt{-4}$ is not a real number. Why?

To find the square root of a number, ask what number squared equals that number.
What squared equals 49? What multiplied by itself equals 49? $7 \cdot 7 = (7)^2 = 49$ so $\sqrt{49} = 7$

The $\sqrt{}$ symbol represents ONLY the positive square root $\sqrt{49} = 7$ even though $(-7)^2 = 49$.

TRY: $\sqrt{64}$ $\qquad -\sqrt{25}$ $\qquad \sqrt{-9}$

What number cubed equals 27?

What multiplied by itself three times equals 27?

$3 \cdot 3 \cdot 3 = (3)^3 = 27$ or $\sqrt[3]{27} = 3$

$\sqrt[3]{27}$ is stated "The third root of 27" or "the cube root of 27".

TRY: $\sqrt[3]{8}$ $\sqrt[3]{-1}$ $\sqrt[3]{-125}$

Sometimes, the radicand is a rational expression. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$

TRY: $\sqrt{\frac{36}{100}}$ $-\sqrt{\frac{49}{25}}$

Approximate Roots

Sometimes, the result isn't an integer.

What is $\sqrt{29}$? Since $\sqrt{29}$ is between $\sqrt{25}$ and $\sqrt{36}$, the answer must be between 5 and 6. Since 29 is closer to 25, the answer will be closer to 5 than to 6.

What is $\sqrt{7}$?

Higher Roots

Sometimes, the index is higher than 3.

What number times itself four times is 16? $\sqrt[4]{16}$

Use the chart to find: $\sqrt[5]{1024}$ $\sqrt[4]{625}$ $\sqrt[3]{216}$

Root & Power Chart

$\sqrt[n]{x}=b$	b=2	b=3	b=4	b=5	b=6	b=7	b=8	b=9	b=10	b=11	b=12	b=13	b=14	b=15
n=2	x=4	9	16	25	36	49	64	81	100	121	144	169	196	225
n=3	8	27	64	125	216				1000					
n=4	16	81	256	625					10000					
n=5	32	243	1024											

TRY:

$\sqrt[5]{-1024}$ $\sqrt[4]{-81}$

$\sqrt[3]{216}$ $-\sqrt[4]{16}$ $\sqrt[4]{-16}$

Exponent: 1/n

Rational Exponents

Consider the following:

$\sqrt{4} = \sqrt{2^2} = 2^1 = 2$ $\qquad$ $\sqrt[3]{8} = \sqrt[3]{2^3} = 2^1 = 2$

$\sqrt{16} = \sqrt{2^4} = 2^2 = 4$

$\sqrt{64} = \sqrt{2^6}$ or $\sqrt{64} = 8$ or $\sqrt{2^6} = 2^3$

$\sqrt[3]{64} = \sqrt[3]{2^6}$ or $\sqrt[3]{64} = 4$ or $\sqrt[3]{2^6} = 2^2$

What is the relationship between the power (the exponent) and the index (the root) and the exponent of the answer?

What is $\sqrt{4^6}$ = _____ $\qquad$ What is $\sqrt[3]{10^{12}}$ = _____

What have you discovered? $\sqrt[n]{x^m} = x^{m/n} = x^{\frac{power}{root}}$

The numerator (m) indicates the **power** to which the base is to be raised, and
The denominator (n) indicates the **index** [**root** to be taken].

If the rational exponent can be reduced, do so. $\sqrt[4]{3^8} = 3^{8/4} = 3^2 = 9$

Definition of $a^{1/n}$

If *n* is any positive integer, then $a^{1/n} = \sqrt[n]{a}$, provided that $\sqrt[n]{a}$ is a real number.

$256^{1/4} = \sqrt[4]{256} = 4$ $\qquad$ $(-8)^{1/3} = \sqrt[3]{-8} = -2$ $\qquad$ $\left(\frac{81}{16}\right)^{1/4} = \sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$

TRY:

Rewrite as a radical and evaluate.

$16^{1/2}$ $\qquad$ $(-32)^{1/5}$ $\qquad$ $\left(\frac{27}{125}\right)^{1/3}$

Exponent: m/n

Definitions of $a^{m/n}$ **and** $(a^{1/n})^m$

If *m* and *n* are positive integers and $a^{1/n}$ is a real number, then

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n} \quad \text{or, equivalently,} \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

One can evaluate the power before taking the root. $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

One can rewrite as a radical raised to a power,
then evaluate the radical before evaluating the power.

$$1000^{2/3} = \sqrt[3]{1000^2} = (\sqrt[3]{1000})^2 = 10^2 = 100$$

$$\left(\frac{9}{16}\right)^{3/2} = \left(\sqrt{\frac{9}{16}}\right)^3 = \frac{\sqrt{9}^3}{\sqrt{16}^3} = \frac{3^3}{4^3} = \frac{27}{64}$$

TRY:

Rewrite as a radical and evaluate.

$$-32^{2/5}$$

$$81^{3/4}$$

$$\left(\frac{27}{8}\right)^{4/3}$$

Negative Exponent

Remember the rule for negative integer exponents: $a^{-n} = \dfrac{1}{a^n}$

For negative rational exponents when $a \neq 0$, $a^{-m/n} = \left(\dfrac{1}{a}\right)^{m/n} = \dfrac{1}{a^{m/n}} = \dfrac{1}{(\sqrt[n]{a})^m}$

Notice how the exponent becomes positive on the reciprocal of the base.

$$a^{-2/3} = \left(\frac{1}{a}\right)^{2/3} = \frac{1}{a^{2/3}} = \frac{1}{(\sqrt[3]{a^2})} \; or \; \frac{1}{(\sqrt[3]{a})^2}$$

$$125^{-1/3} = \left(\frac{1}{125}\right)^{1/3} = \frac{1}{125^{1/3}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$$

$$8^{-2/3} = \left(\frac{1}{8}\right)^{2/3} = \frac{1}{8^{2/3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$$

TRY:

Rewrite as a radical and evaluate.

$16^{-3/4}$

$25^{-3/2}$

$(-8)^{-4/3}$

One applies the rule in the same way (using the reciprocal of the base) if a is a rational expression as in:

$$\left(\frac{27}{1000}\right)^{-1/3}=\left(\frac{1000}{27}\right)^{1/3}=\sqrt[3]{\frac{1000}{27}}=\frac{10}{3}$$

$$-\left(\frac{1024}{243}\right)^{-3/5}=-\left(\frac{243}{1024}\right)^{3/5}=-\left(\sqrt[5]{\frac{243}{1024}}\right)^{3}=-\left(\frac{3}{4}\right)^{3}=-\frac{27}{64}$$

TRY:

Rewrite as a radical and evaluate.

$$\left(\frac{27}{8}\right)^{-1/3}$$

$$\left(\frac{125}{27}\right)^{-2/3}$$

When powers are raised to negative powers, it may be easier to work with the exponents first:

$$\left(\frac{y^{-6}}{8x^{3/7}}\right)^{-2/3}=\frac{y^{-6(-\frac{2}{3})}}{8^{-\frac{2}{3}}x^{\frac{3}{7}(-\frac{2}{3})}}=\frac{y^{4}}{8^{-\frac{2}{3}}x^{-\frac{2}{7}}}=8^{\frac{2}{3}}x^{\frac{2}{7}}y^{4}=\left(\sqrt[3]{8}\right)^{2}x^{\frac{2}{7}}y^{4}=4x^{\frac{2}{7}}y^{4}$$

Exponent Rule Review

Review of Rules for Rational Exponents

Product rule	$a^m a^n = a^{m+n}$
Quotient rule	$a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}, a \neq 0$
Power of a power rule	$(a^m)^n = a^{mn}$
Power of a product rule	$(ab)^n = a^n b^n$
Power of a quotient rule	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, b \neq 0$
Negative exponents rule	$a^{-n} = \dfrac{1}{a^n}, a \neq 0$

The rules for rational exponents follow the rules for integer exponents.

TRY:

Use the rules to evaluate the following. All numbers should be simplified. Do not leave any answers with radical signs. Be sure all exponents are positive.

$y^{1/3} y^{1/3}$ $\quad 2^{1/2} 2^{1/3}$ $\quad 5^{1/4} 5^{-1/4}$ $\quad (a^{1/2} b^{-1/3})(ab)$

$(3^{10})^{1/5}$ $\quad (125a^8)^{1/3}$ $\quad \left(\dfrac{2a^{1/2}}{b^{1/3}}\right)^6$ $\quad (-27x^9)^{1/3}$

$\left(\dfrac{a^{-1/2}}{3a^{2/3}}\right)^{-3}$ $\quad (tv^{1/3})^2 (t^2 v^{-3})^{-1/2}$

Converting to Exponential Form

Convert to exponential form then simplify.

$\sqrt[8]{y^4}$

$\sqrt[3]{6^3}$

$\sqrt[10]{x^8}$

$(\sqrt{5})^2$

For future reference: Square Root of x^2 **:** $(x^2)^{1/2} = |x|$ for any real number x

For some applications, one may need to note the results using absolute value when taking the even root of an even power.