Lesson 17: Solving Rational Equations

Rational Equations

Rational Equation
An equation that contains a rational expression is called a **rational equation**.

Know the difference!

Rational Expression: $\frac{3}{x}-\frac{1}{5}$	Rational Equation: $\frac{3}{x}-\frac{1}{5}=0$
One **evaluates** the **expression**.	One **solves** the **equation**.
When working with a Rational Expression, one **maintains** the denominator by converting every term of the equation to an equivalent fraction of the LCD.	When working with a Rational Equation, one **eliminates** the denominator by multiplying every term of the equation by the LCD.
The **result** is a simplified expression. $\frac{15-x}{5x}$	The **solution**, x=15, is shown in a solution set. { 15 }

Solving a Rational EQUATION

1. Find the LCD of **all** rational expressions in the equation.

2. Eliminate the denominators by multiplying each term of **both** sides of the equation by the LCD of the rational expressions in the equation.

3. Use the procedure for solving linear equations to solve the resulting equation.

4. **Check** the solution to see that it makes the rational equation true. Take extra care to check your solution when a variable appears in the denominator. Because equations involving rational expressions have variables in denominators, a solution (root) to the equation might cause a 0 to appear in the denominator making the rational expression undefined. In this case, the solution (root) does not satisfy the original equation, and so it is called an **extraneous root**.

Example:

$$\frac{12}{3x^2+12x}=1-\frac{1}{x+4}$$

- If the equation involves three or more rational expressions being added or subtracted, first check to see if any fraction can be reduced, and do so.
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FACTOR first $\frac{3 \bullet 4}{3x(x+4)}=1-\frac{1}{x+4}$ then reduce

- Find the LCD (lowest common denominator).

$\frac{4}{x(x+4)}=1-\frac{1}{x+4}$ The LCD is $x(x+4)$.

- Multiply **every** term by the LCD: $\frac{4}{x(x+4)}\cdot(x)(x+4)=1\cdot(x)(x+4)-\frac{1}{x+4}\cdot(x)(x+4)$

- If you have the correct LCD, all denominators will cancel out.

$$4=1(x)(x+4)-1(x)$$

- Continue to solve.

$4=x^2+4x-x$ Combine like terms and move everything to one side.

$0=x^2+3x-4$ Factor: $0=(x+4)(x-1)$ Either $0=x+4$ or $0=x-1$

that is $x=-4$ or $x=1$; **Possible** Solutions: -4 or 1

NOW CHECK to see if -4 and 1 can be in the domain.

-4 makes the denominator of the first fraction undefined;
1 is okay.

Solution: $\{1\}$

Example:

$$\frac{5}{6x}-\frac{1}{8x}=\frac{17}{24}$$

Find the LCD:

$\frac{5}{6x}-\frac{1}{8x}=\frac{17}{24}$	6x	2 3 x
	8x	2 x 2 2
	LCD:	2 3 x 2 2 = 24x

$$\frac{5}{6x}-\frac{1}{8x}=\frac{17}{24} \rightarrow \frac{5\cdot 24x}{6x}-\frac{1\cdot 24x}{8x}=\frac{17\cdot 24x}{24} \rightarrow 5\cdot 4-1\cdot 3=17\cdot x \rightarrow 20-3=17x$$

$$\rightarrow 17=17x \rightarrow \frac{17}{17}=\frac{17x}{17} \rightarrow 1=x$$

Is 1 part of the Domain? Yes.

Therefore the solution is: $\{1\}$

Example:

$$\frac{x-2}{x-6}-\frac{4}{x}=\frac{24}{x^2-6x}$$

Find the LCD:

$\frac{x-2}{x-6}-\frac{4}{x}=\frac{24}{x^2-6x}$	$x-6$	$(x-6)$
	x	x
	x^2-6x	$(x-6)$ x
	LCD:	$(x-6)$ x = $x\,(x-6)$

Be sure the denominators are factored first.

$$\frac{x-2}{x-6}-\frac{4}{x}=\frac{24}{x(x-6)} \rightarrow \frac{(x-2)(x)(x-6)}{x-6}-\frac{4(x)(x-6)}{x}=\frac{24(x)(x-6)}{(x)(x-6)} \rightarrow$$

$$(x-2)(x)-4(x-6)=24 \rightarrow x^2-2x-4x+24=24 \rightarrow x^2-6x=0$$

$x(x-6)=0 \rightarrow$ $x=0$ or $(x-6)=0 \rightarrow$ $x=0$ or $x=6$

Are 0 and 6 part of the Domain? Will either of them make a denominator in the equation 0?

Both values are excluded from the domain. Therefore, there is no solution. $\varnothing$

TRY:

Find the solution set to each equation.

$$\frac{3}{x}+\frac{1}{5}=\frac{1}{2}$$

$$\frac{5}{x-1}+\frac{1}{2x}=\frac{1}{x}$$

$$\frac{x-3}{x+2}=3-\frac{1-2x}{x+2}$$

Rational Equations: Proportions

Proportions

An equation that expresses the equality of two rational expressions is called a **proportion**.

Extremes–Means Property. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. This equation is a proportion.
In it, a and d are called the **extremes**; b and c are called the **means**.

$\frac{5}{x} = \frac{7}{9}$ can be written $5 \cdot 9 = 7x$ $45 = 7x$ $\frac{45}{7} = x$ Solution set: $\left\{ \frac{45}{7} \right\}$

Examples:

$\frac{3}{x-2} = \frac{x+2}{7}$ ➔ $3(7) = (x+2)(x-2)$ ➔

$21 = x^2 - 4$ ➔ $0 = x^2 - 25$ ➔ $0 = (x+5)(x-5)$ ➔

$x+5=0$ or $x-5=0$ ➔ $x=-5$ or $x=5$

Are -5 and 5 part of the Domain? Yes. Solution: $\{-5, 5\}$

- If an equation involves only two rational expressions, make sure one is on each side of the equal sign and use the **Extremes-Means** technique (that is, cross multiply to form an equation to solve).

$\frac{3}{2x-5} + \frac{2}{2x+3} = 0$ Move one fraction to the other side (Don't forget the sign change.)

$\frac{3}{2x-5} = \frac{-2}{2x+3}$ Use the Means-Extremes technique to solve.

$3(2x+3) = -2(2x-5)$ ➔ $6x+9 = -4x+10$ ➔ $6x+4x = 10-9$

Combine terms $10x = 1$ ➔ $\frac{10x}{10} = \frac{1}{10}$ ➔ $x = \frac{1}{10}$

Check the domain. The possible solution is ok. Solution: $\left\{ \frac{1}{10} \right\}$

- If an equation involves three rational expressions AND it is easy to combine two in order to make the equation proportion, do so. Then use the **Extremes-Means** technique to solve.

$$\frac{5}{2w+6}-\frac{1}{w+3}=\frac{1}{w-1}$$

If the middle rational expression were multiplied by $\frac{2}{2}$, it would have the same denominator as the first fraction and could easily be combined with the first one.

This would result in: $\frac{5}{2w+6}-\frac{2}{2w+6}=\frac{1}{w-1}$ ➔

$\frac{3}{2w+6}=\frac{1}{w-1}$ ➔ $3(w-1)=2w+6$ ➔

$3w-3=2w+6$ ➔ $w=9$

Check the domain or the ORIGINAL equation. The possible solution is ok. Solution: $\{9\}$

This method may be more time consuming. It is offered only as an alternative.

TRY:

Find the solution set to each equation.

$$\frac{3}{8}=\frac{5}{x}$$

$$\frac{a-5}{a+6}=\frac{a-7}{a+8}$$

$$\frac{7}{3x-9}-\frac{1}{x-3}=\frac{4}{9}$$

Rational Equations: Solving for Variables

Solving for a variable – isolate the variable in this rational expression.

Solve for m_1 in: $F = k\dfrac{m_1 m_2}{r^2}$

$$F = k\frac{m_1 m_2}{r^2} \Rightarrow F \cdot r^2 = k\frac{m_1 m_2}{r^2} \cdot r^2 \Rightarrow Fr^2 = km_1 m_2 \Rightarrow \frac{Fr^2}{km_2} = \frac{km_1 m_2}{km_2} \Rightarrow \frac{Fr^2}{km_2} = m_1$$

TRY:

Solve for y in: $\dfrac{y-h}{x-k} = a$

Rational Equations: Proportion Problems

(Examples of how to set up the problem are included here. The actual process of arriving at the solution to the problem is often left to you.)

Proportion Problems (use means and extremes)

1. If a number is subtracted from the numerator of $\frac{13}{8}$ and added to the denominator of $\frac{13}{8}$, the resulting fraction is equivalent to $\frac{2}{5}$. Find the number.

   Unknown: x = number

   Equation: $\frac{13-x}{8+x}=\frac{2}{5}$

2. In a sample of 24 returned videotapes, it was found that only 3 were rewound as requested. If 872 videos are returned in a day, then how many of them would you expect to find that are **not** rewound?

   Unknowns: N = number of videos rewound
   872 –N = number of videos not rewound

   Equation: $\frac{rewound}{returned}=\frac{3}{24}=\frac{N}{872}$

TRY:

1. If a number is added to the numerator of $\frac{12}{41}$ and twice the number is added to the denominator of $\frac{12}{41}$, the resulting fraction is equivalent to $\frac{1}{3}$. Find the number.

2. The ratio of pickups to cars sold at a dealership is 2 to 3. If the dealership sold 142 more cars than pickups in 1999, then how many of each did it sell?

Rational Equations: Distance=Rate*Time Problems

Distance = rate • time or time = distance / rate or rate = distance / time

1. B can drive 600 miles in the same time as it takes K to drive 500miles. If B drives 10 mph faster than K, then how fast does B drive?

   Unknowns: B = B's speed, K = K's speed
   Equations: $B = K + 10$

   $$B.time = K.time \qquad B.time = \frac{B.dist}{B} \qquad K.time = \frac{K.dist}{K}$$

   So $\frac{B.dist}{B} = \frac{K.dist}{K}$ $\quad \frac{600}{K+10} = \frac{500}{K}$

   $600K = 500(K+10)$ Once K is found, be sure to solve for B to answer the question.

2. The speed of Lazy River's current is 5 mph. If a boat travels 20 miles downstream in the same time that it takes to travel 10 miles upstream, what is the speed of the boat in still water?

   When the time is the same, the equation for the still-water speed of a boat with (downstream) or against (upstream) a current and for the still-air speed of a plane traveling with (a tailwind) or against (into) the wind (current) is:

   $$\frac{Downstream}{Still + current} = \frac{Upstream}{Still - current}$$

   Unknown: $S = Still$ Equation: $\frac{20}{S+5} = \frac{10}{S-5}$

3. A small jet has airspeed (rate in still air) of 300 mph. One day the co-pilot noted that the plane traveled 85 mph with a tailwind in the same time it took to travel 65 miles against the same wind. What was the rate of the wind?

   Unknown: $W = wind$ Equation: $\frac{85}{300+W} = \frac{65}{300-W}$

TRY:

1. Norma can row her boat 12 miles in the same time as it takes Marietta to cover 36 miles in her motorboat. If Marietta's boat travels 15 mph faster than Norma's boat, then how fast is Norma rowing her boat?

2. The new DogJet flew 500 miles with a steady 25 mph tailwind in the same amount of time it took to fly 400 miles against the same wind. What was the plane’s airspeed (rate in still air)?

3. A boat can travel 16 mph in still water. If the boat can travel 5 mi downstream in the same time it takes to travel 3 miles upstream, what is the speed of the current?

Rational Equations: Work Problems

Work Rate:
If a job can be completed in t units of time, then the rate of work is 1/t of the job per unit of time.

Work Equation:
If T represents the time to complete the job, and t_1 represents the amount of time necessary for unit 1 (person1) to complete the job, and t_2 represents the amount of time necessary for unit 2 (person2) to complete the job, and so on, then to find out how long it would take all the units (all the people) working together to complete the job, solve the following equation for T:

$1/t_1 + 1/t_2 + \ldots = 1/T$ or $(1/t_1)T + (1/t_2)T + \ldots = (1/T)T$
Multiplying each part by T, results in 1 job completed.

1. Mickey can clean the game room in 20 minutes. Terry can clean the game room in 30 minutes. Find how long it will take them to clean the game room if they work together.

Method 1 set up – what does it take to accomplish 1complete job?

$\frac{1}{20}T + \frac{1}{30}T = 1$ LCD is 60 $\frac{1(60)}{20}T + \frac{1(60)}{30}T = 1(60)$ $3T + 2T = 60$ $5T = 60$

$T = 12$ minutes

Method 2 set up – what portion of the total job will be completed in one unit of time? For example, if it takes 20 minutes to do the job, then 1/20 of the job is completed in 1 minute and 1/T of the job is completed in 1 minute. [Personally, I like to use this method. You may use either.]

$\frac{1}{20} + \frac{1}{30} = \frac{1}{T}$ LCD is 60T $\frac{1(60T)}{20} + \frac{1(60T)}{30} = \frac{1(60T)}{T}$ $3T + 2T = 60$ $5T = 60$

$T = 12$ minutes

2. The old printing press took twice as long as the new press to print newspapers. Together it takes 12 hours to print the papers. How long does it take the old press to do the job alone?

   T = new press time 2T = old press time

   $\frac{1}{T} + \frac{1}{2T} = \frac{1}{12}$ (using Method 2)

3. Using the green hose one can fill a small pond in 45 minutes. If both the blue and the green hose are used, one can fill the same pond in 20 minutes. How long would it take to fill the pond using only the blue hose?

   T = Blue hose time $\frac{1}{45} + \frac{1}{T} = \frac{1}{20}$(using Method 2)

TRY:

1. Three postal workers can sort a stack of mail in 20 minutes, 30 minutes, and 60 minutes, respectively. How long will it take them to sort the mail if all three work together?

2. It takes Alan twice as long to prepare the lab beakers as Steve. With Steve's help, the beakers can be processed in 4 hours. How long does it take Steve to prepare the beakers alone?

3. Julie can paint a fence by herself in 12 hours. With Betsy's help, it takes only 5 hours. How long would it take Betsy by herself?