# Lesson 16: Rational Expressions

## Rational Expressions: Domain

**Rational Expression**

A **rational expression** is any algebraic expression that can be written as a quotient of two polynomials where the denominator is not equal to zero.

$\frac{3}{7}$ (ratio of 2 monomials)     $z+6$ (understood denominator of 1)

Consider: $\frac{x+3}{y-7}$ When is this fraction undefined?

$y \neq 7$ because 7 will cause the denominator to be 0.

Therefore, a restriction is placed on the variable in the denominator. One can use any number for $y$ except 7.

> **Restrictions** on any variable in the denominator are found by setting all factors of the denominator, that contain the variable, equal to zero and solving the resulting equation(s) for the variable. Values of the variable that make the denominator zero are ***excluded*** from the domain.
>
> Therefore, the **domain** of a rational expression is the set of all replacement values of the variable for which the expression is **defined**. Stated another way, the domain of a rational expression is the set of all real numbers except the value(s) of the variable that make(s) the denominator equal 0.

In $\frac{x+3}{y-7}$, if $y=7$ then the denominator would be 0.

The domain may be stated two ways:

$\{y \mid y \neq 7\}$ in set-builder notation or $(-\infty, 7) \cup (7, \infty)$ in interval notation

 

**Finding the Domain of a Rational Expression**

1. Completely factor the denominator.

2. Set each factor of the denominator containing the variable equal to zero (using the zero product property where "given real numbers p and q, if p·q = 0, then p = 0 or q = 0).

3. Solve the resulting equations. The solutions are the restrictions placed on the variable. The solutions are *excluded* from the domain.

State the **domain** in set-builder and interval notation.

$$\frac{3b+1}{b^2-3b}$$

Factors of the denominator $(b)(b-3)$

$\{b \mid b \neq 0, b \neq 3\}$ in set-builder notation

$(-\infty,0)\cup(0,3)\cup(3,\infty)$ in interval notation

$$\frac{y+5}{y^2+9}$$

Denominator will not factor.

$y^2$ will always be positive

$\{y \mid y \in \mathrm{Re}\,als\}$ $(-\infty,\infty)$

TRY:

$$\frac{2y-1}{y^2-9}$$

# Rational Expressions: Lowest Terms

**Basic Principle of Rational Numbers**

If $\frac{a}{b}$ is a rational number and $c$ is a nonzero real number, then $\frac{a}{b}=\frac{a}{b}\cdot\frac{c}{c}$ and $\frac{a}{b}\cdot\frac{c}{c}=\frac{a}{b}$

That is, both the numerator and the denominator of a rational expression may be multiplied or divided by the same nonzero polynomial without changing the value of the expression.

**To Write a Rational Expression in Lowest Terms**

1. Completely factor the numerator and the denominator.

2. Divide the numerator and the denominator by the greatest common factor. (One may need to factor out a common factor with a negative sign to get identical factors in the numerator and the denominator.)

$$\frac{x^2+5x}{xy+5y}=\frac{x(x+5)}{y(x+5)}=\frac{x}{y}\cdot\frac{(x+5)}{(x+5)}=\frac{x}{y}\cdot 1=\frac{x}{y}$$

CAUTION: $\frac{x+5}{y+5}\neq\frac{x}{y}$ One can ONLY reduce factors!

TRY:

Simplify each of the following rational expressions.

$$\frac{3a+3}{3} \qquad \frac{7x-14}{7x} \qquad \frac{b^8-ab^5}{ab^5} \qquad \frac{2a^2-2b^2}{2a^2+2b^2}$$

 

# Simplifying with Opposite Factors

In general, for all real numbers a and b, a ≠ b,

$(a-b)=-1(b-a)$ and $\dfrac{a-b}{b-a}=\dfrac{-1(b-a)}{b-a}=-1$

Ex: $(5-b)=-1(b-5)$ $\dfrac{5-b}{b-5}=\dfrac{-1(b-5)}{(b-5)}=-1$

TRY: $\dfrac{5b-10a}{2a-b}$

Sometimes many steps are involved:

$\dfrac{3m^2-3m+m-1}{3-3m}$

$\dfrac{3m(m-1)+1(m-1)}{3(1-m)}$ factor the numerator by grouping and factor out the GCF in the denominator

$\dfrac{(m-1)(3m+1)}{3(1-m)}$ finish factoring the numerator

$\dfrac{(m-1)(3m+1)}{-3(m-1)}$ change $3(1-m)$ to $-3(m-1)$ by factoring out a -1

$\dfrac{(3m+1)}{-3}$ reduce by $(m-1)$

$-\dfrac{3m+1}{3}$ write the negative sign out in front of the expression

# Rational Function: Domain

**Domain of a Rational Function**

The domain of a rational function is the same as the domain of the rational expression used to define the function.

When working with rational functions, one must first be sure the value given to use in the function is part of the domain. Think carefully about the domains of each of the following rational expressions.

Find the indicated value for each given rational expression if the value given is part of the domain.

$N(x) = \dfrac{x+3}{x^3 - 2x^2 - 2x - 3}$ Find $N(3)$

If the denominator cannot be easily factored, try the value and see if it makes the denominator 0.

TRY:

$T(x) = \dfrac{5-x}{x-5}$ Find $T(-9)$

$G(a) = \dfrac{3-5a}{2a+7}$ Find $G(5)$

# Rational Expressions: Multiplication

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

To Multiply Rational Expressions

1. Completely factor the numerators and the denominators where possible.
2. Divide both numerator and denominator by all common factors.
3. Multiply the remaining factors in the numerator and in the denominator.

$$\frac{5x-5y}{x}\cdot\frac{1}{x-y} = \frac{5(x-y)}{x}\cdot\frac{1}{x-y} = \frac{5}{x}\cdot\frac{1}{1} = \frac{5}{x}$$

TRY:

$$\frac{15}{4x^2}\cdot\frac{12x^4}{5}$$

$$\frac{(x-5)(x+4)}{12(x-3)}\cdot\frac{6(x-3)}{(x+4)(x-5)}$$

$$\frac{2x(x-7)}{(x+5)(x-7)}\cdot\frac{(x+6)(x+5)}{3(x+6)}$$

$$\frac{19x^2}{12y-1}\cdot\frac{1-12y}{3x}$$

$$\frac{3a-3y}{3a-3y-ab+by}\cdot\frac{b^2-9}{6b+18}$$

$$\frac{x^2+5x+6}{x}\cdot\frac{x^2}{3x+6}\cdot\frac{9}{x^2-4}$$

# Rational Expressions: Division

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

When dividing two rational expressions, the quotient is obtained by multiplying the first rational expression by the reciprocal of the second rational expression.

$$\frac{15x^3}{4} \div \frac{5x^2}{8} = \frac{15x^3}{4} \cdot \frac{8}{5x^2} = \frac{3x}{1} \cdot \frac{2}{1} = 6x$$

$$\frac{1}{3-m} \div \frac{1}{2m-6} = \frac{1}{3-m} \cdot \frac{2m-6}{1} = \frac{1}{3-m} \cdot \frac{2(m-3)}{1} = \frac{2(m-3)}{3-m} = \frac{2(m-3)}{-1(m-3)} = \frac{2}{-1} = -2$$

$$\frac{x^2+6x+9}{18} \div \frac{(x+3)^2}{36} = \frac{x^2+6x+9}{18} \cdot \frac{36}{(x+3)^2} = \frac{(x+3)(x+3)}{18} \cdot \frac{36}{(x+3)(x+3)} = \frac{36}{18} = 2$$

| | | | |
|---|---|---|---|
| Sometimes the division problem is given in the form of a complex fraction. | $\dfrac{\frac{x^2-36}{x^4}}{\frac{3x-18}{2x^3}}$ | To solve this type of problem, think of the problem as the top fraction divided by the bottom fraction. Then rewrite the problem as the top fraction multiplied by the reciprocal of the bottom fraction and solve. | $\frac{x^2-36}{x^4} \div \frac{3x-18}{2x^3}$ <br> $\frac{x^2-36}{x^4} \cdot \frac{2x^3}{3x-18}$ |
| $\frac{x^2-36}{x^4} \cdot \frac{2x^3}{3x-18} = \frac{(x-6)(x-6)}{x^4} \cdot \frac{2x^3}{3(x-6)} = \frac{2(x-6)}{3x}$ or $\frac{2x-12}{3x}$ | | | |

TRY:

$$\frac{3x^2+3}{5} \div \frac{3x+3}{5}$$

$$10 \div \frac{a+b}{5}$$

$$\frac{2x^2-5x-12}{6+4x} \div \frac{x^2-16}{2}$$

$$\frac{\frac{4x-3}{x}}{\frac{20x-15}{2x^2}}$$

# Addition and Subtraction Properties

Remember how one adds or subtracts two fractions with the same denominator:

$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$ $\quad\frac{6}{7}-\frac{2}{7}=\frac{6-2}{7}=\frac{4}{7}$ The same process works with rational expressions.

**Property of Adding or Subtracting Rational Expressions**

If $b\neq 0$, then $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ and $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$

To add or subtract two rational expressions having the same denominators, add or subtract the numerators and place this sum or difference over the common denominator.

$$\frac{9-4y}{3y}+\frac{6-y}{3y}=\frac{9-4y+6-y}{3y}=\frac{15-5y}{3y}$$

$$\frac{9-4y}{3y}-\frac{6-y}{3y}=\frac{9-4y-(6-y)}{3y}=\frac{9-4y-6+y}{3y}=\frac{-3y+3}{3y}=\frac{-3(y-1)}{3y}=-\frac{y-1}{y}$$

TRY:

$\frac{8x-5}{7x+3}+\frac{x-1}{7x+3}$ $\qquad\qquad\frac{3z-7}{z(z-2)}-\frac{-2z+3}{z(z-2)}$

Remember how one adds or subtracts two fractions with different denominators?
Both fractions must be first changed to equivalent fractions with the same denominator.

Add $\frac{1}{6}+\frac{3}{10}$. First, one needs to find the LCM of 6 and 10. 2·3=6 2·5=10

The factors of the LCM are: 2·3·5 which equals 30.

Change each fraction into an equivalent fraction: $\frac{1}{6}\cdot\frac{5}{5}+\frac{3}{10}\cdot\frac{3}{3}$

Add the equivalent fractions: $\frac{5}{30}+\frac{9}{30}=\frac{14}{30}=\frac{7}{15}$

A similar process works for rational expressions.

# Rational Expressions: Least Common Denominator

**Finding the Least Common Denominator** ( the least expression that is divisible by the denominator of each of the rational expressions ) **for Polynomials**

1. Factor each denominator polynomial completely. Use exponents to express repeated factors.
2. Write the product of all of the different factors that appear in the polynomials.
3. For each factor, use the highest power of that factor in any of the polynomials.

Use a Factor-tree table. Find the LCD for $\frac{2}{15x^2}+\frac{5}{12x}$ :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\frac{2}{15x^2}$ | 3 | 5 | X | X | | | |
| $\frac{5}{12x}$ | 3 | | X | | 2 | 2 | |
| LCD: | 3 | 5 | X | X | 2 | 2 | = $60x^2$ |

Use a Factor-tree table. Find the LCD for $\frac{1}{x^2+5x+6}+\frac{1}{x^2+6x+9}$:

| | | | | |
|---|---|---|---|---|
| $\frac{1}{x^2+5x+6}$ | $(x+2)$ | $(x+3)$ | | |
| $\frac{1}{x^2+6x+9}$ | | $(x+3)$ | $(x+3)$ | |
| LCD: | $(x+2)$ | $(x+3)$ | $(x+3)$ | = $(x+2)(x+3)^2$ |

Try:

Use a Factor-tree table. Find the LCD for $\frac{1}{z^2-25}+\frac{1}{5z+25}+\frac{1}{5z-25}$:

| | | | | |
|---|---|---|---|---|
| $\frac{1}{z^2-25}$ | | | | |
| $\frac{1}{5z+25}$ | | | | |
| $\frac{1}{5z-25}$ | | | | |
| LCD: | | | | |

# Rational Expressions: Different Denominators

Addition and Subtraction of Rational Expressions Having Different Denominators

1. Find the LCD of the rational expression.
2. Write each rational expression as an equivalent rational expression with the LCD as the denominator.
3. Perform the indicated addition or subtraction as before.
4. Reduce the result to lowest terms.

$\frac{2}{w^2-16}+\frac{3}{4w+16}-\frac{7}{4w-16}$ Factor the denominators. $\frac{2}{(w+4)(w-4)}+\frac{3}{4(w+4)}-\frac{7}{4(w-4)}$

The LCD is: $4(w+4)(w-4)$. Multiply each fraction (top and bottom) by the missing factor of the LCD.

$$\frac{2(4)}{(4)(w+4)(w-4)}+\frac{3(w-4)}{4(w+4)(w-4)}-\frac{7(w+4)}{4(w-4)(w+4)}$$

Add or subtract the resulting expression.

$$\frac{8}{(4)(w+4)(w-4)}+\frac{3w-12}{4(w+4)(w-4)}-\frac{7w+28}{4(w-4)(w+4)}$$

$$\frac{8+3w-12-(7w+28)}{(4)(w+4)(w-4)}=\frac{8+3w-12-7w-28}{(4)(w+4)(w-4)}=\frac{-4w-32}{(4)(w+4)(w-4)}$$

ALWAYS check to see if the final rational expression can be factored and reduced.

$\frac{-4w-32}{(4)(w+4)(w-4)}=\frac{-4(w+8)}{(4)(w+4)(w-4)}=\frac{-(w+8)}{(w+4)(w-4)}$ usually written $-\frac{w+8}{(w+4)(w-4)}$

One may leave the denominator in factored form.

Sometimes the denominators cannot be factored, so the LCD is the product of the denominators.

$$\frac{2}{2x-3}+\frac{5}{x+4}=\frac{2(x+4)}{(2x-3)(x+4)}+\frac{5(2x-3)}{(x+4)(2x-3)}=$$

$$\frac{2(x+4)+5(2x-3)}{(2x-3)(x+4)}=\frac{2x+8+10x-15}{(2x-3)(x+4)}=\frac{12x-7}{(2x-3)(x+4)}$$

Sometimes the original rational expressions can be factored and reduced, thus making a smaller LCD.

$\frac{8r}{2r^2+4r+2}-\frac{3r-3}{r^2-1}$ = $\frac{8r}{2(r+1)(r+1)}-\frac{3(r-1)}{(r-1)(r+1)}$ = $\frac{4r}{(r+1)(r+1)}-\frac{3}{(r+1)}$ LCD = $(r+1)(r+1)$

$\frac{4r}{(r+1)(r+1)}-\frac{3(r+1)}{(r+1)(r+1)}$ Only the denominator of the last rational expression needed to be changed.

$\frac{4r-3(r+1)}{(r+1)(r+1)}$= $\frac{4r-3r-3}{(r+1)(r+1)}$= $\frac{r-3}{(r+1)(r+1)}$or $\frac{r-3}{(r+1)^2}$

TRY:

$$\frac{2}{15x^2}+\frac{5}{12x}$$

$$\frac{5x^2}{30xy}-\frac{30x}{80y}$$

$$\frac{2}{a^2b}-\frac{3}{ab^2}$$

$$\frac{5}{x+2}+\frac{3}{x-2}$$

$$\frac{3}{x-5}+\frac{7}{5-x}$$

$$\frac{1}{x^2+5x+6}-\frac{1}{x^2+6x+9}$$

$$\frac{2}{z^2-25}+\frac{2}{5z+25}-\frac{2}{5z-25}$$

# Complex Fractions

A rational expression containing one or more fractions in the numerator, the denominator, or both is called a **complex fraction**.

To simplify a complex fraction, first combine the expressions in the numerator to form one fraction and combine the expressions in the denominator to form one fraction.

Rewrite the problem as the rational expression in the numerator divided by the rational expression in the denominator.

Then write the problem as the first expression multiplied by the reciprocal of the second expression.

Multiply, then simplify if possible.

$$\frac{\frac{1}{x}-\frac{1}{y}}{1-\frac{x}{y}} = \frac{\frac{1\cdot y}{xy}-\frac{1\cdot x}{xy}}{\frac{1\cdot y}{1\cdot y}-\frac{x}{y}} = \frac{\frac{y-x}{xy}}{\frac{y-x}{y}} = \frac{y-x}{xy} \div \frac{y-x}{y} = \frac{y-x}{xy} \cdot \frac{y}{y-x} = \frac{(y-x)y}{xy(y-x)} = \frac{1}{x}$$

TRY:

$$\frac{\frac{x^2+1}{x}}{y+\frac{1}{y}}$$

# Caution with Rational Expressions

## Thoughts to Think when approaching Rational Expressions including Cautions about Common Errors

Looking at a single fraction

- Factor the numerator if possible; factor the denominator if possible; reduce – cancel only the factors
  $\frac{2-4x}{3-4x} \neq \frac{2}{3}$ one CANNOT cancel the $-4x$ because $-4x$ is not a factor!
    - $\frac{2-4x}{6-8x} = \frac{2(1-2x)}{2(3-4x)} = \frac{1-2x}{3-4x}$ one CAN cancel the 2's. In the final answer, one CANNOT reduce $-2x$ and $-4x$ because <u>they are not factors</u>!
    - $\frac{21x^3}{35ax^2} = \frac{3\cdot 7\cdot x\cdot x\cdot x}{5\cdot 7\cdot a\cdot x\cdot x} = \frac{3x}{5a}$ single terms are composed of factors
    - $\frac{x^2-x-12}{x^2+5x+6} = \frac{(x+3)(x-4)}{(x+3)(x+2)} = \frac{x-4}{x+2}$ One CANNOT cancel out the xs or 2s
    - 

Looking at two rational terms (meaning two fractions)

- If you are to MULTIPLY or DIVIDE two fractions, process each rational term (fraction) following the steps above. Then…
    - To multiply, after reducing each individual fraction, look to see if anything in either numerator cancels with anything in either denominator. If so, reduce. Multiply the remaining factors together for the final answer.
      $$\frac{2x^2+5x-12}{x^2+2x}\cdot\frac{x^3-3x^2}{2x^2-9x+9} = \frac{(2x-3)(x+4)}{x(x+2)}\cdot\frac{x^2(x-3)}{(2x-3)(x-3)} = \frac{x(x+4)}{(x+2)}$$
    - To divide, multiply by the reciprocal of the second fraction (turn it upside down), then reduce each individual fraction as much as possible. After reducing each individual fraction, look to see if anything in either numerator cancels with anything in either denominator. If so, reduce. Multiply the remaining factors together for the final answer.
      $$\frac{2x^2+5x-12}{x^2+2x}\div\frac{x+3}{x-3} = \frac{(x+2)(x+3)}{x(x+2)}\cdot\frac{(x-3)}{(x+3)} = \frac{(x+3)}{x}\cdot\frac{(x-3)}{(x+3)} = \frac{x-3}{x}$$

- To ADD or SUBTRACT two fractions:

  - IF the two denominators are already the same, just combine the numerators, then reduce.

    $\frac{9-4y}{3y}+\frac{6-y}{3y}=\frac{15-5y}{3y}$ One cannot reduce by 3 or by y as these are NOT factors of both the numerator and the denominator. Be careful when SUBTRACTING two fractions. Distribute the subtraction sign (a negative) to all terms in the numerator of the second fraction. Notice the 3 cannot be canceled until it is factored out first.

    $$\frac{9+4y}{3y}-\frac{3+y}{3y}=\frac{9+4y-3-y}{3y}=\frac{6+3y}{3y}=\frac{3(2+y)}{3y}=\frac{2+y}{y}$$

  - IF the two denominators are NOT the same, reduce each fraction as much as possible. Determine the LCD that must be used. Multiply each numerator by the factors in the LCD that are NOT in that fraction's denominator. Combine all numerators OVER the LCD – the denominator. Factor the combined numerator to see if you can reduce further.

    $\frac{2}{x^2-9}+\frac{3}{x^2-6x+9}=\frac{2}{(x+3)(x-3)}+\frac{3}{(x-3)(x-3)}=$ LCD is $(x+3)(x-3)^2$

    $$\frac{2(x-3)+3(x+3)}{(x+3)(x-3)^2}=\frac{2x-6+3x+9}{(x+3)(x-3)^2}=\frac{5x+3}{(x+3)(x-3)^2}$$

When two factors are opposite in signs:

- Factor out a -1 in one of the factors. $\frac{a-5}{5-a}=\frac{-1(5-a)}{5-a}=-1$ BE VERY CAREFUL to check to see if you have factored correctly. Sometimes, if the two terms within the expression are switched around, it may appear necessary to remove a -1, but it isn't. After factoring, reduce is possible.

  $$\frac{-7y+7x}{x}\bullet\frac{1}{x-y}=\frac{7x-7y}{x}\bullet\frac{1}{x-y}=\frac{7(x-y)}{x(x-y)}=\frac{7}{x}$$

  (No need to factor out a -1.)