Lesson 14: Working with Polynomials

Polynomials: An Introduction

Vocabulary

Example: $x^2 + 4x - 9$

Term — A single number or the product of a number and one or more variables raised to **whole number** powers. $x^2 + 4x - 9$ has 3 terms.

Constant — A number that **does not change** its value; -9 is called a constant term

Numerical coefficient — A number **preceding** the variable in each term; understood to be 1 if none appears. 1 for the 1st term and 4 for the 2nd term

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Polynomial — A single term or the sum of a finite number of **terms** with **positive integer** exponents on the variables

1. It has **real** number coefficients.
2. All variables in a polynomial are raised only to **positive integer** powers.
3. The operations performed by the variables are **limited** to addition, subtraction, and multiplication.

Monomial — Polynomial with just one term 7 x^2 $7x^2$

Binomial — Polynomial with two terms $4x + 9$

Trinomial — Polynomial with three terms $x^2 + 4x - 9$

Degree of a term — The **sum** of the exponents on the variables $x^2 y^2$ is degree 4

Degree of a polynomial — The **highest** degree of any nonzero term of the polynomial

Note: a nonzero constant has degree 0. Ex: The constant 5 has degree 0 since one can think of 5 as $5 \cdot 1 = 5 \cdot x^0$

The number 0 has no degree since 0 times a variable to any power is 0.

Standard form — Arrange the polynomial in **decreasing** powers of the variable.

For each of the following, state whether the expression is a polynomial or not.

If it is a polynomial, identify the type of polynomial (based on the number of terms), and state the degree of the polynomial.

$x^3+3x^4-5x^6$	-18	$4x^2y-2xy^3$	$\frac{4}{3x^2}$	$\frac{3x^3}{5}$
Trinomial, degree 6	monomial, degree 0	binomial, degree 4	Not a polynomial	monomial, degree 3

TRY: $3x+5$ 15 $4x^{-2}$ $5x^3-x^2y^2$

Value of the Polynomial: Depends on the value given to the variable.

Example: In x^2+4x-9 , if $x=3$, then the value of x^2+4x-9 is $(3)^2+4(3)-9$ or 12

TRY: What is the value of $3x^2+4x$ when $x=-2$?

What is the value of x^3+4 when $x=2$?

Polynomials are often identified with capital italicized letters.

P(x) is read “P of x.” Q(x) is read “Q of x.”

P(x) is used to denote the value of the polynomial
when the value in the ( ) is substituted throughout the polynomial for x.

Example:

Let P(x) = $3x^2 + 4x - 2$

What is P(5) ? This is shorthand for asking the question,

What is the value of the polynomial when x=5 ?

P(5) = 3 $(5)^2$ + 4(5) – 2
P(5) = 3 (25) + 20 – 2
P(5) = 75 + 20 – 2
P(5) = 95 – 2
P(5) = 93

TRY:

Let $P(x) = 2x^2 + 3x - 1$ and $Q(x) = x^2 - 4x + 3$

Find: P(3)

P(-2)

P(-2) – Q(-1)

Find: Q(2)

Q(-1)

P(5) ÷ Q(0)

<u>Polynomial Function</u>

Polynomial Function P(x) = $x^2 - x - 2$ is called a polynomial function since $x^2 - x - 2$ is a polynomial.

$P(x) = x^2 - x - 2$, find $P(-1)$

$$P(x) = x^2 - x - 2$$
$$P(-1) = (-1)^2 - (-1) - 2$$
$$P(-1) = 1 + 1 - 2$$
$$P(-1) = 0$$

The value of P(x) is a function of the value for x used in the polynomial $x^2 - x - 2$.

Addition and Subtraction of Polynomials

ADDITION of POLYNOMIALS

Simplifying expressions is **combining like** terms.

Note: x is the same as $1x$ and $-x$ is the same as $-1x$

The process of addition or subtraction is **performed only with the numerical coefficients** of like terms (same variable and power). The variable factors remain unchanged.

$$5y^2 - 12y + 6y^5 - 4y^2 + y = 6y^5 + y^2 - 11y$$

TRY:

$6a^2b - 2ab^2 + 3a^2b - 4ab^2$

$2y^2 - 3y - 8 + y + 4y - 1$

SUBTRACTION of POLYNOMIALS

To subtract polynomials use the Distributive Property to remove grouping symbols.
Be careful to distribute the "-" to each term [that is multiply each term in the group by a " -1"].

$$(5ab^2 - 2a^2b^2) - (4a^2b^2 + 3ab^2) = 5ab^2 - 2a^2b^2 - 4a^2b^2 - 3ab^2 = -6a^2b^2 + 2ab^2$$

TRY:

$5x - (-3x + 4)$

$3a - (12a + 4b)$

$(a - 3a) - (1 - a - 2a^2)$

$(2x - 5) - (x^2 - 3x + 2)$

Multiplication of Polynomials (Monomials, Binomials, FOIL)

Extended Distributive Property $a(b_1+b_2+\ldots+b_n) = ab_1+ab_2+\ldots+ab_n$

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.

$$-a^3b(3a^2b^5 - ab^4 - 7a^2b) = -3a^5b^6 + a^4b^5 + 7a^5b^2$$

$$[x^3 - 4x(x^2 - 3x + 2) - 5x] + [x^2 - 5(4 - x^2) + 3] =$$

$$[x^3 - 4x^3 + 12x^2 - 8x - 5x] + [x^2 - 20 + 5x^2 + 3] =$$

$$[-3x^3 + 12x^2 - 13x] + [6x^2 - 17] =$$

$$-3x^3 + 18x^2 - 13x - 17$$

TRY:

$$-1(-x^2 - 3x - 9)$$

$$-3x(x-2) - 5[2x - 4(x+6)]$$

$$-3mn(2mn^2 - 4mn - 9)$$

$$(-4xy)(3x^2y - 5xy)$$

To multiply two polynomials, multiply each term of the first polynomial by each term of the second polynomial and then combine like terms.

$$(3x+y)(4x-y)=3x(4x)+3x(-y)+y(4x)+y(-y)=12x^2-3xy+4xy-y^2=12x^2+xy-y^2$$

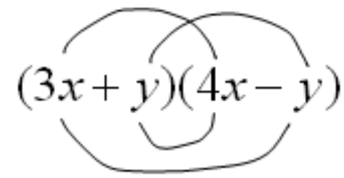
Mr. Foil

F – First terms	3x · 4x	$12x^2$
O – Outer terms	3x · -y	$-3xy$
I – Inner terms	y · 4x	$+4xy$
L – Last terms	y · -y	$-y^2$

$12x^2-3xy+4xy-y^2$

$= \quad 12x^2+xy-y^2$

Do you see Mr. Foil?



Some like to use a table:

*	3x	+y
4x	$12x^2$	$+4xy$
-y	$-3xy$	$-y^2$

Multiply each term in the row by a term in the column. Add the resulting 4 terms, combining like terms. $12x^2-3xy+4xy-y^2$ = $12x^2+xy-y^2$

TRY:

$(x+7)(x-8)$ $\qquad$ $(4a-7b)(3a-5b)$ $\qquad$ $(2x^3+3)(x^2-3)$

What happens when you multiply these?

$(x-6)(x+6)$ $\qquad$ $(3a+2b)(3a-2b)$ $\qquad$ $(x^2+3)(x^2-3)$

Multiplication of Larger or Three Polynomials and Higher Powers

To multiply two larger polynomials, one might use vertical multiplication. ($2a^2-3a+5$)(a^2-4a+2)

$$\begin{array}{rl} & 2a^2-3a+5 \\ \times & a^2-4a+2 \\ \hline & 4a^2-6a+10 \\ & -8a^3+12a^2-20a \\ & 2a^4-3a^3+5a^2 \end{array}$$

multiply 2 times $2a^2-3a+5$

multiply $-4a$ times $2a^2-3a+5$

multiply a^2 times $2a^2-3a+5$

Answer: $2a^4-11a^3+21a^2-26a+10$ add like terms together

One can also approach the problem by using the distributive property. Multiply each term of the 2nd polynomial by each term of the 1st polynomial. Decide which method works best for you.

TRY:

$(a^2+a+b)(a^2-a-b)$ $(x^2-3x+2)(x-4)$

To multiply three polynomials, first multiply two together, then take that product times the third polynomial.

$$(x-2y)(x+2y)(3x-y)=(x^2+2xy-2yx-4y^2)(3x-y)=(x^2-4y^2)(3x-y)$$
$$(x^2-4y^2)(3x-y)=3x^3-x^2y-12xy^2+4y^3$$

A similar process would be used for higher powers or for multiplying many polynomials together.

Higher Powers

$(x+5)^4=(x+5)(x+5)(x+5)(x+5)$

Multiply the first two binomials together. Then, take that product times the third polynomial. Then take that product times the fourth polynomial.

TRY:

$(x-2y)(2x+y)(x-y)$

Special Binomial Products

Square of a Binomial: In general for all real numbers a and b,

Square of a Sum: $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$ NOT $a^2 + b^2$!

Example: $(3y+4)^2 = (3y+4)(3y+4) = 9y^2 + 2(3y \cdot 4) + 16 = 9y^2 + 24y + 16$

Square of a Difference: $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$ NOT $a^2 - b^2$!

Example: $(3y-4)^2 = (3y-4)(3y-4) = 9y^2 - 2(3y \cdot 4) + 16 = 9y^2 - 24y + 16$

What happens when you multiply these?

$(x-6)(x+6)$ $(3a+2b)(3a-2b)$ $(x^2+3)(x^2-3)$

Product of a Sum & Difference: $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$
difference of squares

Example: $(3y+4)(3y-4) = 9y^2 - 12y + 12y - 16 = 9y^2 - 16$

$(3y+4)(3y-4) = (3y)^2 - (4)^2 = 9y^2 - 16$

TRY: $(2x+3)^2$

$(2x-3)^2$

$(x+3)(x-3)$

To multiply something like:

$[(x+y)+2][(x+y)-2]$, think of this product of a sum and difference.

$$[(x+y)+2][(x+y)-2] = (x+y)^2 - 4 = x^2 + 2xy + y^2 - 4$$

While it may be useful and more efficient to memorize these special products, one can always arrive at the correct answer by using FOIL.

Challenge: $(2y^t - 3)(4y^t + 7)$

Division of Polynomials

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial and simplify.

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \quad \text{when} \quad c \neq 0$$

To divide $\frac{15x^2-25x+5x}{5x}$, think of it as $\frac{15x^3}{5x}-\frac{25x^2}{5x}+\frac{5x}{5x}$

Simplify each fraction to get $3x^2-5x+1$. (Be careful. 5x divided by itself is 1.)

TRY: $$\frac{12x^2+8x-16}{4}$$

$$\frac{20x^2+30x}{10x}$$

To divide a polynomial by a binomial, use long division.

To divide: $$\frac{3x^2+19x+20}{x+5}$$

1. Write the problem as long division with the numerator as the dividend and the denominator as the divisor.
2. Think – what does one need to multiply the first term of the divisor by in order to obtain the first term of the dividend – what does one multiply x by to get $3x^2$? 3x
3. Place the answer over the correct column in the dividend. In this case, place the 3x over the 19x.

$$
\begin{array}{r}
3x + 4 \\
x+5 \overline{\smash{)}\,3x^2 + 19x + 20} \\
-(3x^2 + 15x) \\
\hline
4x + 20 \\
-(4x + 20) \\
\hline
0
\end{array}
$$

4. Multiply the 3x times the binomial x+5 and place the result under the dividend.

5. Put parentheses around the result and a minus sign in front to indicate subtraction. THIS IS VERY IMPORTANT as it helps one avoid errors.

6. Subtract.

7. Bring down the next term of the dividend.

8. Repeat the process.

9. Check the answer by multiplying the quotient times the divisor to obtain the dividend.

If the problem has a remainder, place it over the divisor to form a fraction.

$$16c^3 - 38c^2 - 11c + 19 \quad \div \quad 2c - 5 \quad = \quad 8c^2 + c - 3 + \frac{4}{2c-5}$$

If the dividend is missing terms of some degrees,

expand the dividend by adding place holders for the missing degrees.

For example, for $x^3 + 125 \div x + 5$ rewrite the dividend as:
$x^3 + 0x^2 + 0x + 125$ before performing long division.

$$x^3 + 0x^2 + 0x + 125 \quad \div \quad x + 5 \quad = \quad x^2 - 5x + 25$$ [Try it.]